

# On Completely Singular von Neumann Subalgebras

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## Abstract

Let  $\mathcal{M}$  be a von Neumann algebra acting on a Hilbert space  $\mathcal{H}$ , and  $\mathcal{N}$  be a singular von Neumann subalgebra of  $\mathcal{M}$ . If  $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$  is singular in  $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$  for every Hilbert space  $\mathcal{K}$ ,  $\mathcal{N}$  is said to be *completely singular* in  $\mathcal{M}$ . We prove that if  $\mathcal{N}$  is a singular abelian von Neumann subalgebra or if  $\mathcal{N}$  is a singular subfactor of a type II<sub>1</sub> factor  $\mathcal{M}$ , then  $\mathcal{N}$  is completely singular in  $\mathcal{M}$ . If  $\mathcal{H}$  is separable, we prove that  $\mathcal{N}$  is completely singular in  $\mathcal{M}$  if and only if for every  $\theta \in \text{Aut}(\mathcal{N}')$  such that  $\theta(X) = X$  for all  $X \in \mathcal{M}'$ , then  $\theta(Y) = Y$  for all  $Y \in \mathcal{N}'$ . As the first application, we prove that if  $\mathcal{M}$  is separable (with separable predual) and  $\mathcal{N}$  is completely singular in  $\mathcal{M}$ , then  $\mathcal{N} \bar{\otimes} \mathbb{L}$  is completely singular in  $\mathcal{M} \bar{\otimes} \mathbb{L}$  for every separable von Neumann algebra  $\mathbb{L}$ . As the second application, we prove that if  $\mathcal{N}_1$  is a singular subfactor of a type II<sub>1</sub> factor  $\mathcal{M}_1$  and  $\mathcal{N}_2$  is a completely singular von Neumann subalgebra of  $\mathcal{M}_2$ , then  $\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$  is completely singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ .

**Keywords:** von Neumann algebras, singular von Neumann subalgebras, completely singular von Neumann subalgebras, tensor products of von Neumann algebras.

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## 1 Introduction

Let  $\mathcal{M}$  be a von Neumann algebra acting on a Hilbert space  $\mathcal{H}$ . A von Neumann subalgebra  $\mathcal{N}$  of  $\mathcal{M}$  is *singular* if the only unitary operators in  $\mathcal{M}$  satisfying the condition  $U\mathcal{N}U^* = \mathcal{N}$  are those in  $\mathcal{N}$ . The study of singular von Neumann subalgebras has a long and rich history (see for instance [1, 6, 8, 9, 11]). Recently, there is a remarkable progress on singular MASAs (maximal abelian von Neumann subalgebras) in type II<sub>1</sub> factors (see [13, 12, 14]). In [13], Allan Sinclair and Roger Smith introduced a concept of asymptotic homomorphism property. In [12], a concept of weak asymptotic homomorphism property is introduced. Let  $\mathcal{M}$  be a type II<sub>1</sub> factor and  $\mathcal{N}$  be a von Neumann subalgebra of  $\mathcal{M}$ . Then  $\mathcal{N} \subseteq \mathcal{M}$

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has the weak asymptotic homomorphism property if for all  $X_1, \dots, X_n \in \mathcal{M}$  and  $\epsilon > 0$ , there exists a unitary operator  $U \in \mathcal{N}$  such that

$$\|E_{\mathcal{N}}(X_i U X_j^*) - E_{\mathcal{N}}(E_{\mathcal{N}}(X_i) U E_{\mathcal{N}}(X_j)^*)\|_2 < \epsilon.$$

Remarkably, in [14], it was shown that every singular MASA in a type  $\text{II}_1$  factor satisfies the weak asymptotic homomorphism property. As a corollary, the tensor product of singular MASAs in type  $\text{II}_1$  factors is proved to be a singular MASA in the tensor product of type  $\text{II}_1$  factors (see [14]), which is a well-known hard question for long time.

It is very natural to ask the following question: if  $\mathcal{N}_1$  and  $\mathcal{N}_2$  are singular von Neumann subalgebras of  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , respectively, is  $\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$  singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ ? It turns out this is not always true. Let  $\mathcal{M}_1 = M_3(\mathbb{C})$  and  $\mathcal{N}_1 = M_2(\mathbb{C}) \oplus \mathbb{C}$ . Then  $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  are central projections in  $\mathcal{N}_1$  and  $\mathcal{N}_1 = \{P, Q\}'$ . Suppose  $U \in \mathcal{M}_1$  is a unitary matrix such that  $U\mathcal{N}_1 U^* = \mathcal{N}_1$ . Then  $UPU^* = P$  and  $UQU^* = Q$  (because  $adU$  preserves the center of  $\mathcal{N}_1$  and  $\tau(P) = \frac{2}{3}, \tau(Q) = \frac{1}{3}$ , where  $\tau$  is the normalized trace on  $M_3(\mathbb{C})$ ). So  $U \in \{P, Q\}' = \mathcal{N}_1$ . This implies that  $\mathcal{N}_1$  is singular in  $\mathcal{M}_1$ . Let  $\mathcal{M}_2 = \mathcal{B}(l^2(\mathbb{N}))$  and  $\mathcal{N}_2 = \mathcal{M}_2$ . Then  $\mathcal{N}_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})) = M_2(\mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})) \oplus \mathbb{C} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is not singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})) = M_3(\mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ . Indeed, let  $V$  be an isometry from  $l^2(\mathbb{N})$  onto  $\mathbb{C}^2 \otimes l^2(\mathbb{N})$ , then  $U = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$  is a unitary operator in  $M_3(\mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  such that  $U(\mathcal{N}_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})))U^* = \mathcal{N}_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ . Since  $U$  is not in  $\mathcal{N}_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ ,  $\mathcal{N}_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is not singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ . Indeed,  $\mathcal{N}_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is regular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  (see Remark 2.14).

Let  $\mathcal{M}$  be a von Neumann algebra and  $\mathcal{N}$  be a von Neumann subalgebra of  $\mathcal{M}$ . If  $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$  is singular in  $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$  for every Hilbert space  $\mathcal{K}$ , then  $\mathcal{N}$  is said to be *completely singular* in  $\mathcal{M}$ . In section 2, we prove that if  $\mathcal{N}$  is a singular MASA or if  $\mathcal{N}$  is a singular subfactor of a type  $\text{II}_1$  factor  $\mathcal{M}$ , then  $\mathcal{N}$  is completely singular in  $\mathcal{M}$ . For every type  $\text{II}_1$  factor  $\mathcal{M}$ , we construct a singular von Neumann subalgebra  $\mathcal{N}$  of  $\mathcal{M}$  ( $\mathcal{N} \neq \mathcal{M}$ ) such that  $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is regular in  $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ . Motivated by Lemma 1.2 of [3], we obtain a nice characterization of complete singularity in section 3. As the first application, in section 4.1, we prove that if  $\mathcal{M}$  is separable and  $\mathcal{N}$  is completely singular in  $\mathcal{M}$ , then  $\mathcal{N} \bar{\otimes} \mathbb{L}$  is completely singular in  $\mathcal{M} \bar{\otimes} \mathbb{L}$  for every separable von Neumann algebra  $\mathbb{L}$ . As the second application, we prove that if  $\mathcal{N}_1$  is a singular subfactor of a type  $\text{II}_1$  factor  $\mathcal{M}_1$  and  $\mathcal{N}_2$  is a completely singular von Neumann subalgebra of  $\mathcal{M}_2$ , then  $\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$  is singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ . The following question seems to be interesting: if  $\mathcal{N}_1, \mathcal{N}_2$  are completely singular von Neumann subalgebras of  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , is  $\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$  completely singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ ?

## 2 On singularity and complete singularity

### 2.1 Normalizer and Normalizing groupoid of $\mathcal{N}$ in $\mathcal{M}$

Let  $\mathcal{M}$  be a von Neumann algebra and  $\mathcal{N}$  be a von Neumann subalgebra of  $\mathcal{M}$ . Then  $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})$  denotes the normalizer of  $\mathcal{N}$  in  $\mathcal{M}$ :

$$\mathfrak{N}_{\mathcal{M}}(\mathcal{N}) = \{U \in \mathcal{M} : U \text{ is a unitary operator, } U\mathcal{N}U^* = \mathcal{N}\},$$

and  $\mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$  denotes the normalizing groupoid of  $\mathcal{N}$  in  $\mathcal{M}$ :

$$\mathfrak{GN}_{\mathcal{M}}(\mathcal{N}) = \{V \in \mathcal{M} : V \text{ is a partial isometry with initial space } E \text{ and final space } F$$

$$\text{such that } E, F \in \mathcal{N} \text{ and } V\mathcal{N}_E V^* = \mathcal{N}_F\},$$

where  $\mathcal{N}_E = E\mathcal{N}E$  and  $\mathcal{N}_F = F\mathcal{N}F$ .  $\mathcal{N}$  is singular in  $\mathcal{M}$  if and only if  $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$ , the von Neumann algebra generated by  $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ , is  $\mathcal{N}$ . Recall that  $\mathcal{N}$  is *regular* in  $\mathcal{M}$  if  $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' = \mathcal{M}$ .

If  $\mathcal{M}$  is a finite von Neumann algebra and  $\mathcal{N}$  is a maximal abelian von Neumann subalgebra of  $\mathcal{M}$ , then  $V \in \mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$  if and only if there is a unitary operator  $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$  and a projection  $E \in \mathcal{N}$  such that  $V = UE$  ([6], Theorem 2.1). In other words: any partial isometry that normalizes  $\mathcal{N}$  extends to a unitary operator that normalizes  $\mathcal{N}$ . As a corollary, we have  $\mathfrak{GN}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$ , i.e., the von Neumann algebra generated by the normalizing groupoid of  $\mathcal{N}$  in  $\mathcal{M}$  is the von Neumann algebra generated by the normalizer of  $\mathcal{N}$  in  $\mathcal{M}$ . If  $\mathcal{M}$  is an infinite factor, e.g., type III, and  $\mathcal{N} = \mathcal{M}$ , then there is an isometry in  $\mathcal{M}$  which can not be extended to a unitary operator in  $\mathcal{M}$ . The following example tells us that even the weak form  $\mathfrak{GN}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$  can fail. Let  $\mathcal{M} = M_3(\mathbb{C})$  and  $\mathcal{N} = M_2(\mathbb{C}) \oplus \mathbb{C}$ . As we point out in the introduction,  $\mathcal{N}$  is singular in  $\mathcal{M}$ , i.e.,

$$\mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' = \mathcal{N}. \text{ Simple computations show that } V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ is in } \mathfrak{GN}_{\mathcal{M}}(\mathcal{N}). \text{ Note}$$

that  $V$  is not in  $\mathcal{N}$ .

Let  $V_1, V_2 \in \mathcal{M}$  be two partial isometries in  $\mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$  and  $E_i = V_i^* V_i \in \mathcal{N}$ ,  $i = 1, 2$ . We say  $V_1 \preceq V_2$  if  $E_1 \leq E_2$  and  $V_1 = V_2 E_1$ . It is obvious that  $\preceq$  is a partial order on the set of partial isometries in  $\mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$ . Let  $\{V_\alpha\}$  be a totally ordered subset of  $\mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$ , then  $V = \lim_\alpha V_\alpha$  (in strongly operator topology) exists and  $V \in \mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$ .

**Lemma 2.1.** *If  $\mathcal{M}$  is a finite von Neumann algebra and  $\mathcal{N}$  is a subfactor of  $\mathcal{M}$ , then for every  $V \in \mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$ , there is a unitary operator  $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$  such that  $V \preceq U$ . In particular,  $\mathfrak{GN}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$ .*

*Proof.* By Zorn's lemma, there is a maximal element  $W \in \mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$  such that  $V \preceq W$ . Let  $E = W^*W$  and  $F = WW^*$ . Then  $E, F \neq 0$  and  $E, F \in \mathcal{N}$ . We need to prove  $E = I$ . If  $E \neq I$ , then  $F \neq I$  since  $\mathcal{M}$  is finite. So  $I - E, I - F \in \mathcal{N}$  are not 0. Since  $\mathcal{N}$  is a factor, there is a partial isometry  $V_1 \in \mathcal{N}$  with initial space  $E_1$ , a non-zero subprojection of  $I - E$ , and final space  $E_2$ , a non-zero subprojection of  $E$ . Let  $F'$  be the range space of  $WE_2$ . Then  $F' = WE_2W^* \in \mathcal{N}$ . Since  $\mathcal{N}$  is a factor, there is a partial isometry  $V_2 \in \mathcal{N}$  with initial space  $F_2$ , a non-zero subprojection of  $F'$ , and final space  $F_1$ , a non-zero subprojection of  $I - F$ . Now  $W' = V_2WV_1$  is a partial isometry with initial space  $E_1 \leq I - E$  and final space  $F_1 \leq I - F$ . Simple computation shows that  $W + W' \in \mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$ . Note that  $V \preceq W \preceq W + W'$  and  $W \neq W + W'$ . It contradicts to the maximality of  $W$ .  $\square$

**Lemma 2.2.** *Let  $\mathcal{M}$  be a von Neumann algebra and  $\mathcal{N}$  be an abelian von Neumann subalgebra of  $\mathcal{M}$ . Then  $\mathfrak{GN}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$ .*

*Proof.* Let  $\mathcal{M}_1 = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$ . We only need to prove that  $\mathfrak{GN}_{\mathcal{M}}(\mathcal{N})'' \subseteq \mathcal{M}_1$ . For  $V \in \mathcal{M}$  a partial isometry, define  $\mathcal{S}(V) = \{W \in \mathcal{M}_1 : W \text{ is a partial isometry and } W \preceq V\}$ . Suppose  $V \notin \mathcal{M}_1$ . By Zorn's lemma, we can choose a maximal element  $W \in \mathcal{S}(V)$  such that  $V - W \neq 0$  and  $\mathcal{S}(V - W) = \{0\}$ . Since  $W \in \mathcal{M}_1$ ,  $V \in \mathcal{M}_1$  if and only if  $V - W \in \mathcal{M}_1$ . Therefore, we can assume that  $V \neq 0$  and  $\mathcal{S}(V) = \{0\}$ . Let  $E = V^*V$  and  $F = VV^*$ . Then  $E \neq 0$  and  $F \neq 0$ .

If  $E = F$ , let  $U = V + (I - E)$ . Then  $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$  and  $V = UE \in \mathcal{M}_1$ . It is a contradiction. If  $E \neq F$ , we can assume that  $E_1 = E(I - F) \neq 0$  (otherwise consider  $V^*$ ). Let  $V_1 = VE_1$  and  $F_1$  be the final space of  $V_1$ . Then  $V_1 \in \mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$  with initial space  $E_1 \leq I - F$  and final space  $F_1 \leq F$  such that  $0 \neq V_1 \preceq V$ . Let  $U = V_1 + V_1^* + (I - E_1 - F_1)$ . Then  $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$  and  $V_1 = UE_1 \in \mathcal{M}_1$ . Note that  $V_1 \neq 0$  and  $V_1 \preceq V$ .  $\mathcal{S}(V) \neq \{0\}$ . It is a contradiction.  $\square$

If  $\mathcal{N}$  is singular in  $\mathcal{M}$  and  $E \in \mathcal{N}$  is a projection,  $\mathcal{N}_E (= E\mathcal{N}E)$  may be not singular in  $\mathcal{M}_E$ . For example, let  $\mathcal{M} = M_3(\mathbb{C})$  and  $\mathcal{N} = M_2(\mathbb{C}) \oplus \mathbb{C}$  and

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{N}.$$

Then  $\mathcal{N}_E$  is not singular in  $\mathcal{M}_E$ . But we have the following result.

**Lemma 2.3.** *Let  $\mathcal{N}$  be a singular von Neumann subalgebra of  $\mathcal{M}$  and  $E \in \mathcal{N}$  be a projection. If  $\mathcal{N}$  is a countably decomposable, properly infinite von Neumann algebra, then  $\mathcal{N}_E$  is singular in  $\mathcal{M}_E$ .*

*Proof.* Let  $P$  be the central support of  $E$  relative to  $\mathcal{N}$ . Then there are central projections  $P_1, P_2$  of  $\mathcal{N}$  such that  $P_1 + P_2 = P$  and  $P_1E$  is finite and  $P_2E$  is properly infinite. Let  $E_1 = P_1E$  and  $E_2 = P_2E$ . Then the central supports of  $E_1$  and  $E_2$  are  $P_1$  and  $P_2$ , respectively. Since  $P_1$  is a properly infinite countably decomposable projection and  $E_1$  is a finite projection in  $\mathcal{N}_{P_1}$  and the central support of  $E_1$  is  $P_1$ ,  $P_1$  is a countably infinite sum of projections  $\{E_{1n}\}$  in  $\mathcal{N}$ , each  $E_{1n}$  is equivalent to  $E_1$  in  $\mathcal{N}_{P_1}$  (see for instance, Corollary 6.3.12 of [7], volume 2). For  $n \in \mathbb{N}$ , let  $W_{1n}$  be a partial isometry in  $\mathcal{N}_{P_1}$  such that  $W_{1n}^*W_{1n} = E_{1n}$  and  $W_{1n}W_{1n}^* = E_1$ . Since  $P_2$  and  $E_2$  are properly infinite projections in  $\mathcal{N}_{P_2}$  with same central support  $P_2$  and  $\mathcal{N}_{P_2}$  is countably decomposable,  $P_2$  and  $E_2$  are equivalent in  $\mathcal{N}_{P_2}$  (see for instance, Corollary 6.3.5 of [7], volume 2). Since  $P_2$  is properly infinite in  $\mathcal{N}$ , it can be decomposed into a countably infinite sum of projections  $\{E_{2n}\}$ , each  $E_{2n}$  is equivalent to  $P_2$  and hence to  $E_2$ . For  $n \in \mathbb{N}$ , let  $W_{2n}$  be a partial isometry in  $\mathcal{N}_{P_2}$  such that  $W_{2n}^*W_{2n} = E_{2n}$  and  $W_{2n}W_{2n}^* = E_2$ . Let  $W_n = W_{1n} + W_{2n} \in \mathcal{N}$ . Then  $W_n^*W_n = E_{1n} + E_{2n}$  and  $W_nW_n^* = E_1 + E_2 = E$ .

Suppose  $V$  is a unitary operator in  $\mathcal{M}_E$  such that  $V\mathcal{N}_E V^* = \mathcal{N}_E$ . Define  $U = \sum_{n=1}^{\infty} W_n^* V W_n + (I - P_1 - P_2)$ . Then  $U$  is a unitary operator and  $U^* = \sum_{n=1}^{\infty} W_n^* V^* W_n + (I - P_1 - P_2)$ . For any  $T \in \mathcal{N}$ ,  $UTU^* = \sum_{m,n=1}^{\infty} W_m^* V W_m T W_n^* V^* W_n + (I - P_1 - P_2)T$ . Note that  $W_m T W_n^* \in \mathcal{N}_E$ ,  $V W_m T W_n^* V^* \in \mathcal{N}_E$ . So  $UTU^* \in \mathcal{N}$ . Similarly,  $U^* T U \in \mathcal{N}$ . Thus  $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ . Since  $\mathcal{N}$  is singular in  $\mathcal{M}$ ,  $U \in \mathcal{N}$ . Therefore,  $W_1^* V W_1 = U(E_{11} + E_{21}) \in \mathcal{N}$ . So  $V = E V E = W_1 W_1^* V W_1 W_1^* \in \mathcal{N}_E$ . This implies that  $\mathcal{N}_E$  is singular in  $\mathcal{M}_E$ .  $\square$

## 2.2 Singular MASA and singular subfactor (of type $\text{II}_1$ factor) are completely singular

**Theorem 2.4.** *Let  $\mathcal{N}$  be a von Neumann subalgebra of  $\mathcal{M}$  and  $\mathcal{K}$  be a Hilbert space. If  $\mathfrak{G}\mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$ , then  $\mathfrak{N}_{\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})}(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' \bar{\otimes} \mathcal{B}(\mathcal{K})$ .*

Combine Theorem 2.4, Lemma 2.1 and Lemma 2.2, we have the following corollaries.

**Corollary 2.5.** *If  $\mathcal{M}$  is a type  $\text{II}_1$  factor and  $\mathcal{N}$  is a singular subfactor of  $\mathcal{M}$ , then  $\mathcal{N}$  is completely singular in  $\mathcal{M}$ .*

**Corollary 2.6.** *If  $\mathcal{N}$  is a singular MASA of a von Neumann algebra  $\mathcal{M}$ , then  $\mathcal{N}$  is completely singular in  $\mathcal{M}$ .*

To prove Theorem 2.4, we need the following lemmas. We consider  $\dim \mathcal{K} = 2$  first, which motivates the general case.

**Lemma 2.7.** *Let  $U = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$  be a unitary operator in  $\mathcal{M} \bar{\otimes} M_2(\mathbb{C})$ . Then the following conditions are equivalent:*

1.  $U(\mathcal{N} \bar{\otimes} M_2(\mathbb{C}))U^* = \mathcal{N} \bar{\otimes} M_2(\mathbb{C});$
2.  $A_i X A_j^* \in \mathcal{N}$  and  $A_i^* X A_j \in \mathcal{N}$  for all  $X \in \mathcal{N}$ ,  $1 \leq i, j \leq 4$ .

*Proof.*  $U(\mathcal{N} \bar{\otimes} M_2(\mathbb{C}))U^* = \mathcal{N} \bar{\otimes} M_2(\mathbb{C})$  if and only if  $U(\mathcal{N} \bar{\otimes} M_2(\mathbb{C}))U^* \subseteq \mathcal{N} \bar{\otimes} M_2(\mathbb{C})$  and  $U^*(\mathcal{N} \bar{\otimes} M_2(\mathbb{C}))U \subseteq \mathcal{N} \bar{\otimes} M_2(\mathbb{C})$ .  $U(\mathcal{N} \bar{\otimes} M_2(\mathbb{C}))U^* \subseteq \mathcal{N} \bar{\otimes} M_2(\mathbb{C})$  if and only if  $U \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U^*, U \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} U^*, U \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} U^*, U \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} U^* \in \mathcal{N}$  for all  $X \in \mathcal{N}$ . Simple computations show that  $U(\mathcal{N} \bar{\otimes} M_2(\mathbb{C}))U^* \subseteq \mathcal{N} \bar{\otimes} M_2(\mathbb{C})$  if and only if  $A_i X A_j^* \in \mathcal{N}$  for all  $X \in \mathcal{N}$ ,  $1 \leq i, j \leq 4$ .  $\square$

Since the proof of the following lemma is similar to the proof of Lemma 2.7, we omit the proof.

**Lemma 2.8.** *Let  $U = (A_{ij})$  be a unitary operator in  $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$ . Then the following conditions are equivalent:*

1.  $U(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))U^* = \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K});$
2.  $A_i X A_j^* \in \mathcal{N}$  and  $A_i^* X A_j \in \mathcal{N}$  for all  $X \in \mathcal{N}$ ,  $1 \leq i, j \leq \dim \mathcal{K}$ .

Let  $X = I$  and  $i = j$  in 2 of Lemma 2.8. We have the following corollary.

**Corollary 2.9.** *Let  $U = (A_{ij})$  be a unitary operator in  $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$  such that  $U(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))U^* = \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ . If  $A_{ij} = V_{ij}H_{ij}$  is the polar decomposition of  $A_{ij}$ , then  $H_{ij} \in \mathcal{N}$ ,  $1 \leq i, j \leq \dim \mathcal{K}$ .*

**Lemma 2.10.** *Let  $\mathcal{N}$  be a von Neumann algebra and  $H$  be a positive operator in  $\mathcal{N}$  and  $E$  be the closure of the range space of  $H$ . Then the strong-operator closure of  $\mathcal{T} = \{H X H : X \in \mathcal{N}\}$  is  $\mathcal{N}_E (= E \mathcal{N} E)$ .*

*Proof.* It is easy to see  $\mathcal{T} \subseteq \mathcal{N}_E$ . Let  $H = \int_{\mathbb{R}} \lambda dE(\lambda)$  and  $E_n = E([1/n, \infty))$ . Then  $\lim_{n \rightarrow \infty} E_n = E$  in strong-operator topology. Set  $H_n = E_n H + (I - E_n)$ . Then  $H_n$  is invertible in  $\mathcal{N}$ . For  $X \in \mathcal{N}_E$ , let  $X_n = H_n^{-1}(E_n X E_n)H_n^{-1} \in \mathcal{N}$ . Then  $H X_n H = H H_n^{-1} E_n X E_n H_n^{-1} H = E_n X E_n \rightarrow E X E = X$  in strong-operator topology. Hence, the strong-operator closure of  $\mathcal{T}$  contains  $\mathcal{N}_E$ .  $\square$

**Lemma 2.11.** *Suppose  $\mathcal{N}$  is a von Neumann subalgebra of  $\mathcal{M}$  and  $A \in \mathcal{M}$  satisfying  $A \mathcal{N} A^* \subseteq \mathcal{N}$  and  $A^* \mathcal{N} A \subseteq \mathcal{N}$ . Let  $A = V H$  be the polar decomposition and  $E = V^* V$ ,  $F = V V^*$ . Then  $H, E, F \in \mathcal{N}$  and  $V \in \mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$ .*

*Proof.* By the assumption,  $A^* I A = H^2 \in \mathcal{N}$ . So  $H \in \mathcal{N}$  and  $E = R(H) \in \mathcal{N}$ , where  $R(H)$  is the closure of range space of  $H$ . By symmetry,  $F \in \mathcal{N}$ . Note that  $A X A^* = V H X H V^* \subseteq F \mathcal{N} F = \mathcal{N}_F$  for all  $X \in \mathcal{N}$ . By lemma 2.10,  $V \mathcal{N}_E V^* \subseteq \mathcal{N}_F$ . By  $A^* X A \subseteq \mathcal{N}$  for all  $X \in \mathcal{N}$  and similar arguments,  $V^* \mathcal{N}_F V \subseteq \mathcal{N}_E$ . So  $\mathcal{N}_F \subseteq V \mathcal{N}_E V^*$ . Thus  $V \mathcal{N}_E V^* = \mathcal{N}_F$ , i.e.,  $V \in \mathfrak{GN}_{\mathcal{M}}(\mathcal{N})$ .  $\square$

*The proof of Theorem 2.4.* Let  $U_1 \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$  and  $V$  be a unitary operator in  $\mathcal{B}(\mathcal{K})$ . Then  $U_1 \otimes V \in \mathfrak{N}_{\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})}(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))$ . So  $\mathfrak{N}_{\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})}(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))'' \supseteq \mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' \bar{\otimes} \mathcal{B}(\mathcal{K})$ .

Let  $U = (A_{ij})$  be a unitary operator in  $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$  such that  $U(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))U^* = \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ . Let  $A_{ij} = V_{ij}H_{ij}$  be the polar decomposition of  $A_{ij}$ . By Lemma 2.8, Corollary 2.9 and Lemma 2.11,  $H_{ij} \in \mathcal{N}$  and  $V_{ij} \in \mathfrak{G}\mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ . By the assumption of Theorem 2.4,  $V_{ij} \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$ . So  $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' \bar{\otimes} \mathcal{B}(\mathcal{K})$ , i.e.,  $\mathfrak{N}_{\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})}(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))'' \subseteq \mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' \bar{\otimes} \mathcal{B}(\mathcal{K})$ .  $\square$

## 2.3 On singular but not completely singular von Neumann subalgebras

**Proposition 2.12.** *If  $\mathcal{N}$  is a singular but not a completely singular von Neumann subalgebra of  $\mathcal{M}$ , then there is a von Neumann subalgebra  $\mathcal{M}_1$  of  $\mathcal{M}$  and a Hilbert space  $\mathcal{K}$  such that  $\mathcal{N} \subsetneq \mathcal{M}_1$ ,  $\mathcal{N}$  is singular in  $\mathcal{M}_1$  and  $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$  is regular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{B}(\mathcal{K})$ .*

*Proof.* Since  $\mathcal{N}$  is not completely singular in  $\mathcal{M}$ , there is a Hilbert space  $\mathcal{K}$  such that  $\mathbb{L} = \mathfrak{N}_{\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})}(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))'' \supsetneq \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ . Since  $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}) \subseteq \mathbb{L} \subseteq \mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$ ,  $\mathbb{L} = \mathcal{M}_1 \bar{\otimes} \mathcal{B}(\mathcal{K})$  for some von Neumann algebra  $\mathcal{M}_1$ ,  $\mathcal{N} \subsetneq \mathcal{M}_1 \subseteq \mathcal{M}$ . Since  $\mathcal{N}$  is singular in  $\mathcal{M}$ ,  $\mathcal{N}$  is singular in  $\mathcal{M}_1$ . Since  $\mathcal{M}_1 \bar{\otimes} \mathcal{B}(\mathcal{K}) = \mathfrak{N}_{\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})}(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))$ ,  $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$  is regular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{B}(\mathcal{K})$ .  $\square$

**Proposition 2.13.** *If  $\mathcal{M}$  is a type  $\text{II}_1$  factor, then there is a singular von Neumann subalgebra  $\mathcal{N}$  of  $\mathcal{M}$  such that  $\mathcal{N} \neq \mathcal{M}$  and  $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is regular in  $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ .*

*Proof.* Let  $\mathcal{M}_1$  be a type  $\text{I}_3$  subfactor of  $\mathcal{M}$  and  $\mathcal{M}_2 = \mathcal{M}'_1 \cap \mathcal{M}$ . Then  $\mathcal{M}_2$  is a type  $\text{II}_1$  factor. We can identify  $\mathcal{M}$  with  $M_3(\mathbb{C}) \bar{\otimes} \mathcal{M}_2$  and  $\mathcal{M}_1$  with  $M_3(\mathbb{C}) \bar{\otimes} \mathbb{C}I$ . With this identification, let  $\mathcal{N} = (M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{M}_2$ . Then

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes I, \text{ and } Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes I$$

are central projections in  $\mathcal{N}$ .  $\mathcal{N} = \{P, Q\}' \cap \mathcal{M}$  and  $\{P, Q\}''$  is the center of  $\mathcal{N}$ . Let  $U \in \mathcal{M}$  be a unitary operator such that  $UNU^* = \mathcal{N}$ . Then  $U\{P, Q\}''U^* = \{P, Q\}''$ . Let  $\tau$  be the unique tracial state on  $\mathcal{M}$ . Then  $\tau(P) = \frac{2}{3}$  and  $\tau(Q) = \frac{1}{3}$ . So  $UPU^* = P$  and  $UQU^* = Q$ . This implies that  $U \in \{P, Q\}' \cap \mathcal{M} = \mathcal{N}$  and  $\mathcal{N}$  is singular in  $\mathcal{M}$ .

To see  $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is not singular in  $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ , we identify  $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  with  $M_3(\mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})) \bar{\otimes} \mathcal{M}_2$  and  $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  with  $(M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})) \bar{\otimes} \mathcal{M}_2$ . Let  $V$  be an isometry from  $l^2(\mathbb{N})$  onto  $\mathbb{C}^2 \otimes l^2(\mathbb{N})$ , then  $U = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$  is a unitary operator

in  $M_3(\mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  such that  $U((M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})))U^* = (M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ . So  $U \otimes I$  is a unitary operator in the normalizer of  $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  but  $U \otimes I \notin \mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ .

By Proposition 2.12, there is a von Neumann subalgebra  $L$  of  $\mathcal{M}$  such that  $\mathcal{N} \subsetneq L$  and  $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is regular in  $L \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ . Since  $(M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{M}_2 \subsetneq L \subseteq M_3(\mathbb{C}) \bar{\otimes} \mathcal{M}_2$ , by Ge-Kadison's splitting theorem (see [4]),  $L = L_1 \bar{\otimes} \mathcal{M}_2$  for some von Neumann algebra  $L_1$  such that  $M_2(\mathbb{C}) \oplus \mathbb{C} \subsetneq L_1 \subseteq M_3(\mathbb{C})$ . Since  $M_3(\mathbb{C})$  is the unique von Neumann subalgebra of  $M_3(\mathbb{C})$  satisfies the above condition,  $L_1 = M_3(\mathbb{C})$ . This implies that  $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is regular in  $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ .  $\square$

**Remark 2.14.** By the proof of Proposition 2.13,  $(M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is regular in  $M_3(\mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ .

**Remark 2.15.** Let  $\mathcal{M}$  be a type  $II_1$  factor and  $\mathcal{N}$  be the singular von Neumann subalgebra constructed as in the proof of Proposition 2.13. It is easy to see that  $\mathcal{N} \bar{\otimes} \mathcal{N}$  is not singular in  $\mathcal{M} \bar{\otimes} \mathcal{M}$ .

### 3 A characterization of complete singularity

**Theorem 3.1.** *Let  $\mathcal{M}$  be a von Neumann algebra acting on a separable Hilbert space  $\mathcal{H}$  and  $\mathcal{N}$  be a von Neumann subalgebra of  $\mathcal{M}$ . Then the following conditions are equivalent.*

1.  $\mathcal{N}$  is completely singular in  $\mathcal{M}$ ;
2.  $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$  is singular in  $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ ;
3. If  $\theta \in \text{Aut}(\mathcal{N}')$  and  $\theta(X) = X$  for all  $X \in \mathcal{M}'$ , then  $\theta(Y) = Y$  for all  $Y \in \mathcal{N}'$ .

*Proof.* “3  $\Rightarrow$  1”. Let  $\mathcal{K}$  be a Hilbert space and  $U \in \mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$  be a unitary operator such that  $U(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))U^* = \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ . Note that  $(\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K}))' = \mathcal{M}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}$  and  $(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))' = \mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}$ .  $\theta = adU \in \text{Aut}(\mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}})$ . Since  $U \in \mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$ ,  $\theta(X \otimes I_{\mathcal{K}}) = X \otimes I_{\mathcal{K}}$  for all  $X \in \mathcal{M}'$ . By the assumption of 3,  $Y \otimes I_{\mathcal{K}} = \theta(Y \otimes I_{\mathcal{K}}) = U(Y \otimes I_{\mathcal{K}})U^*$  for all  $Y \in \mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}$ . This implies that  $U \in \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ . Therefore,  $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$  is singular in  $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$ .

“1  $\Rightarrow$  2” is trivial.

“2  $\Rightarrow$  3”. By [5], there is a separable Hilbert space  $\mathcal{H}_1$  and a faithful normal representation  $\phi$  of  $\mathcal{N}'$  such that  $\phi(\mathcal{N}')$  acts on  $\mathcal{H}_1$  in standard form. Let  $\theta_1 = \phi \cdot \theta \cdot \phi^{-1}$ . Then  $\theta_1 \in \text{Aut} \phi(\mathcal{N}')$  and  $\theta_1(\phi(X)) = \phi(X)$  for all  $X \in \mathcal{M}'$ . Now there is a unitary operator  $U_1 \in \mathcal{B}(\mathcal{H}_1)$  such that  $\theta_1(\phi(Y)) = U_1 \phi(Y) U_1^*$  for all  $Y \in \mathcal{N}'$ . Let  $\mathcal{N}'_1$  and  $\mathcal{M}'_1$  be the



commutant of  $\phi(\mathcal{N}')$  and  $\phi(\mathcal{M}')$  relative to  $\mathcal{H}_1$ , respectively. Then  $\mathcal{N}_1$  is a von Neumann subalgebra of  $\mathcal{M}_1$ . Since  $\theta_1(\phi(X)) = U_1\phi(X)U_1^* = \phi(X)$  for all  $X \in \mathcal{M}'$ ,  $U_1 \in \mathcal{M}_1$ . Since  $\theta = \text{ad}U_1 \in \text{Aut } \phi(\mathcal{N}')$ ,  $\theta = \text{ad}U_1 \in \text{Aut } \mathcal{N}_1$ . Now we only need to prove that  $\mathcal{N}_1$  is a singular von Neumann subalgebra of  $\mathcal{M}_1$ . Then  $U_1 \in \mathcal{N}_1$  and  $\theta_1(\phi(Y)) = U_1\phi(Y)U_1^* = \phi(Y)$  for all  $Y \in \mathcal{N}'$ . This implies that  $\theta(Y) = Y$  for all  $Y \in \mathcal{N}'$ .

By [1] (Theorem 3, page 61),  $\phi = \phi_3 \cdot \phi_2 \cdot \phi_1$ , where  $\phi_1(\mathcal{N}') = \mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}$ ,  $\mathcal{K} = l^2(\mathbb{N})$ ,  $\phi_2(\mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}) = (\mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}})E$ ,  $E \in (\mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}})' = \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$  and  $\phi_3$  is a spacial isomorphism. We may assume that  $\phi_3 = \text{id}$ . Then  $\mathcal{N}_1 = E(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))E$  and  $\mathcal{M}_1 = E(\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K}))E$ , where  $E \in \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ . By 2,  $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$  is singular in  $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$ . Note that  $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$  is a countably decomposable, properly infinite von Neumann algebra. By Lemma 2.3,  $\mathcal{N}_1$  is singular in  $\mathcal{M}_1$ .  $\square$

Note that in the proof of “3  $\Rightarrow$  1” of Theorem 3.1, we do not need the assumption that  $\mathcal{H}$  is a separable Hilbert space.

Let  $\mathcal{M}$  be a von Neumann algebra. A von Neumann subalgebra  $\mathfrak{B}$  of  $\mathcal{M}$  is called *maximal injective* if it is injective and if it is maximal with respect to inclusion in the set of all injective von Neumann subalgebras of  $\mathcal{M}$  (see [9]).

**Proposition 3.2.** *If  $\mathfrak{B}$  is a maximal injective von Neumann subalgebra of  $\mathcal{M}$ , then  $\mathfrak{B}$  is completely singular in  $\mathcal{M}$ .*

*Proof.* We can assume that  $\mathcal{M}$  acts on  $\mathcal{H}$  in standard form. Then  $\mathfrak{B}'$  is a minimal injective von Neumann algebra extension of  $\mathcal{M}'$  (see [3], 1.3). Let  $\theta \in \text{Aut}(\mathfrak{B}')$  such that  $\theta(X) = X$  for all  $X \in \mathcal{M}'$ . Then  $\theta(Y) = Y$  for all  $Y \in \mathfrak{B}'$  by Lemma 1.2 of [3]. By Theorem 3.1,  $\mathfrak{B}$  is completely singular in  $\mathcal{M}$ .  $\square$

## 4 Completely singular von Neumann subalgebras in tensor products of von Neumann algebras

### 4.1

The proof of the following lemma is similar to the proof of Lemma 6.6 of [16]

**Lemma 4.1.** *Let  $\mathcal{M}$  be a separable von Neumann algebra and  $\mathcal{N}$  be a singular von Neumann subalgebra of  $\mathcal{M}$ . If  $\mathcal{A}$  is an abelian von Neumann algebra, then  $\mathcal{N} \bar{\otimes} \mathcal{A}$  is a singular von Neumann subalgebra of  $\mathcal{M} \bar{\otimes} \mathcal{A}$ .*

*Proof.* We can assume that  $\mathcal{M}$  acts on a separable Hilbert space  $\mathcal{H}$  in standard form and  $\mathcal{A}$  is countably decomposable. Then there is a  $*$ -isomorphism from  $\mathcal{A}$  onto  $L^\infty(\Omega, \mu)$  with

$\mu$  a probability Radon measure on some compact space  $\Omega$ . To the  $*$ -isomorphism  $\mathcal{A} \rightarrow L^\infty(\Omega, \mu)$  corresponds canonically a  $*$ -isomorphism  $\Phi$  from  $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{A}$  onto  $L^\infty(\Omega, \mu; \mathcal{B}(\mathcal{H}))$ . Note that  $\Phi(\mathcal{M} \bar{\otimes} \mathcal{A})(\omega) = \mathcal{M}$  and  $\Phi(\mathcal{N} \bar{\otimes} \mathcal{A})(\omega) = \mathcal{N}$  for almost all  $\omega \in \Omega$ . Let  $U \in \mathcal{M} \bar{\otimes} \mathcal{A}$  be a unitary operator such that  $U(\mathcal{N} \bar{\otimes} \mathcal{A})U^* = \mathcal{N} \bar{\otimes} \mathcal{A}$ . Then  $\Phi(U) = U(\omega)$  such that  $U(\omega)$  is a unitary operator in  $\mathcal{M}$  for almost all  $\omega \in \Omega$ . By  $U(\mathcal{N} \bar{\otimes} \mathcal{A})U^* = \mathcal{N} \bar{\otimes} \mathcal{A}$ , we have  $U(\omega)\mathcal{N}U(\omega)^* = \mathcal{N}$  for almost all  $\omega \in \Omega$ . Since  $\mathcal{N}$  is singular in  $\mathcal{M}$ ,  $U(\omega) \in \mathcal{N}$  for almost all  $\omega \in \Omega$ . Hence  $U \in \mathcal{N} \bar{\otimes} \mathcal{A}$ .  $\square$

Since for every Hilbert space  $\mathcal{K}$ ,  $\mathcal{M} \bar{\otimes} \mathcal{A} \bar{\otimes} \mathcal{B}(\mathcal{K})$  is canonically isomorphic to  $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K}) \bar{\otimes} \mathcal{A}$ . We have the following corollary.

**Corollary 4.2.** *Let  $\mathcal{M}$  be a separable von Neumann algebra and  $\mathcal{N}$  be a completely singular von Neumann subalgebra. If  $\mathcal{A}$  is an abelian von Neumann algebra, then  $\mathcal{N} \bar{\otimes} \mathcal{A}$  is a completely singular von Neumann subalgebra of  $\mathcal{M} \bar{\otimes} \mathcal{A}$ .*

**Theorem 4.3.** *Let  $\mathcal{M}$  be a separable von Neumann algebra and  $\mathcal{N}$  be a completely singular von Neumann subalgebra. Then  $\mathcal{N} \bar{\otimes} L$  is completely singular in  $\mathcal{M} \bar{\otimes} L$  for every separable von Neumann algebra  $L$ .*

*Proof.* We can assume that  $\mathcal{M}$  and  $L$  act on separable Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$  in standard form, respectively. Let  $\theta$  be in  $\text{Aut}(\mathcal{N}' \bar{\otimes} L')$  such that  $\theta(X \otimes Z) = X \otimes Z$  for all  $X \in \mathcal{M}'$  and  $Z \in L'$ . Let  $\mathcal{A}$  be the center of  $L'$ . Then  $(\mathbb{C}I_{\mathcal{H}} \bar{\otimes} L')' \cap (\mathcal{N}' \bar{\otimes} L') = (\mathcal{B}(\mathcal{H}) \bar{\otimes} L) \cap (\mathcal{N}' \bar{\otimes} L') = (\mathcal{B}(\mathcal{H}) \cap \mathcal{N}') \bar{\otimes} (L \cap L') = \mathcal{N}' \bar{\otimes} \mathcal{A}$ . So for  $T \in \mathcal{N}' \bar{\otimes} \mathcal{A}$  and  $Z \in L'$ ,  $T(I_{\mathcal{H}} \otimes Z) = (I_{\mathcal{H}} \otimes Z)T$  and  $\theta(T)\theta(I_{\mathcal{H}} \otimes Z) = \theta(I_{\mathcal{H}} \otimes Z)\theta(T)$ . Since  $\theta(I_{\mathcal{H}} \otimes Z) = I_{\mathcal{H}} \otimes Z$ ,  $\theta(T)(I_{\mathcal{H}} \otimes Z) = (I_{\mathcal{H}} \otimes Z)\theta(T)$ . This implies that  $\theta(T) \in \mathcal{N}' \bar{\otimes} \mathcal{A}$ . So  $\theta \in \text{Aut}(\mathcal{N}' \bar{\otimes} \mathcal{A})$  when  $\theta$  is restricted on  $\mathcal{N}' \bar{\otimes} \mathcal{A}$  such that  $\theta(X \otimes Z) = X \otimes Z$  for all  $X \in \mathcal{M}'$  and  $Z \in \mathcal{A}$ .

Consider the standard representation  $\phi$  of  $\mathcal{A}$  on a separable Hilbert space  $\mathcal{K}_1$ . Then  $\phi(\mathcal{A})' = \phi(\mathcal{A})$ . By Corollary 4.2,  $\mathcal{N} \bar{\otimes} \phi(\mathcal{A})$  is completely singular in  $\mathcal{M} \bar{\otimes} \phi(\mathcal{A})$ . On  $\mathcal{H} \otimes \mathcal{K}_1$ ,  $(\mathcal{N} \bar{\otimes} \phi(\mathcal{A}))' = \mathcal{N}' \bar{\otimes} \phi(\mathcal{A})$  and  $(\mathcal{M} \bar{\otimes} \phi(\mathcal{A}))' = \mathcal{M}' \bar{\otimes} \phi(\mathcal{A})$ . Note that  $\theta_1 = (id \bar{\otimes} \phi) \cdot \theta \cdot (id \bar{\otimes} \phi^{-1}) \in \text{Aut}(\mathcal{N}' \bar{\otimes} \phi(\mathcal{A}))$  and  $\theta_1(X \otimes Z') = (id \bar{\otimes} \phi) \cdot \theta(X \otimes \phi^{-1}(Z')) = (id \bar{\otimes} \phi)(X \otimes \phi^{-1}(Z')) = X \otimes Z'$  for all  $X \in \mathcal{M}'$  and  $Z' \in \phi(\mathcal{A})$ . By Theorem 3.1,  $\theta_1(Y \otimes Z') = Y \otimes Z'$  for all  $Y \in \mathcal{M}'$  and  $Z' \in \phi(\mathcal{A})$ . This implies that  $\theta(Y \otimes \phi^{-1}(Z')) = Y \otimes \phi^{-1}(Z')$  for all  $Y \in \mathcal{N}'$  and  $Z' \in \phi(\mathcal{A})$ . Let  $Z' = I_{\mathcal{K}_1}$ . Then  $\theta(Y \otimes I_{\mathcal{K}}) = Y \otimes I_{\mathcal{K}}$  for all  $Y \in \mathcal{N}'$ . Hence  $\theta(Y \otimes Z) = Y \otimes Z$  for all  $Y \in \mathcal{N}'$  and  $Z \in L'$ . By Theorem 3.1,  $\mathcal{N} \bar{\otimes} L$  is completely singular in  $\mathcal{M} \bar{\otimes} L$ .  $\square$

## 4.2

**Theorem 4.4.** *Let  $\mathcal{M}_i$  be a separable von Neumann algebra and  $\mathcal{N}_i$  be a completely singular von Neumann subalgebra of  $\mathcal{M}_i$ ,  $i = 1, 2$ . If  $\mathcal{N}_1$  is a factor, then  $\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$  is completely singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ .*

*Proof.* We can assume that  $\mathcal{M}_1$  and  $\mathcal{M}_2$  act on separable Hilbert space  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in standard form, respectively. Let  $\theta$  be in  $\text{Aut}(\mathcal{N}'_1 \bar{\otimes} \mathcal{N}'_2)$  such that  $\theta(X_1 \otimes X_2) = X_1 \otimes X_2$  for all  $X_1 \in \mathcal{M}'_1$  and  $X_2 \in \mathcal{M}'_2$ .

Since  $\mathcal{N}'_1$  is a singular subfactor in  $\mathcal{M}_1$ ,  $\mathcal{N}'_1 \cap \mathcal{M}_1 = \mathcal{N}'_1 \cap \mathcal{N}_1 = \mathbb{C}I_{\mathcal{H}_1}$ . Note that  $(\mathcal{M}'_1 \bar{\otimes} \mathbb{C}I_{\mathcal{H}_2})' \cap (\mathcal{N}'_1 \bar{\otimes} \mathcal{N}'_2) = (\mathcal{M}_1 \bar{\otimes} \mathcal{B}(\mathcal{H}_2)) \cap (\mathcal{N}'_1 \bar{\otimes} \mathcal{N}'_2) = (\mathcal{M}_1 \cap \mathcal{N}'_1) \bar{\otimes} (\mathcal{B}(\mathcal{H}_2) \cap \mathcal{N}'_2) = \mathbb{C}I_{\mathcal{H}_1} \bar{\otimes} \mathcal{N}'_2$ . We have  $\theta(\mathbb{C}I_{\mathcal{H}_1} \bar{\otimes} \mathcal{N}'_2) = \theta((\mathcal{M}_1 \cap \mathcal{N}'_1) \bar{\otimes} (\mathcal{B}(\mathcal{H}_2) \cap \mathcal{N}'_2)) = \theta((\mathcal{M}_1 \bar{\otimes} \mathcal{B}(\mathcal{H}_2)) \cap (\mathcal{N}'_1 \bar{\otimes} \mathcal{N}'_2)) = \theta((\mathcal{M}'_1 \bar{\otimes} \mathbb{C}I_{\mathcal{H}_2})' \cap (\mathcal{N}'_1 \bar{\otimes} \mathcal{N}'_2)) = \theta(\mathcal{M}'_1 \bar{\otimes} \mathbb{C}I_{\mathcal{H}_2})' \cap \theta(\mathcal{N}'_1 \bar{\otimes} \mathcal{N}'_2) = (\mathcal{M}'_1 \bar{\otimes} \mathbb{C}I_{\mathcal{H}_2})' \cap (\mathcal{N}'_1 \bar{\otimes} \mathcal{N}'_2) = \mathbb{C}I_{\mathcal{H}_1} \bar{\otimes} \mathcal{N}'_2$ . Since  $\mathcal{N}_2$  is completely singular in  $\mathcal{M}_2$  and  $\theta(I_{\mathcal{H}_1} \otimes X_2) = I_{\mathcal{H}_1} \otimes X_2$  for all  $X_2 \in \mathcal{M}'_2$ ,  $\theta(I_{\mathcal{H}_1} \otimes Y_2) = I_{\mathcal{H}_1} \otimes Y_2$  for all  $Y_2 \in \mathcal{M}'_2$  by Theorem 3.1. Therefore,  $\theta(X_1 \otimes Y_2) = X_1 \otimes Y_2$  for all  $X_1 \in \mathcal{M}'_1$  and  $Y_2 \in \mathcal{N}'_2$ . By Theorem 4.3,  $\mathcal{N}'_1 \bar{\otimes} \mathcal{M}_2$  is completely singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ . Since  $\theta(X_1 \otimes Y_2) = X_1 \otimes Y_2$  for all  $X_1 \in \mathcal{M}'_1$  and  $Y_2 \in \mathcal{N}'_2$ , by Theorem 3.1,  $\theta(Y_1 \otimes Y_2) = Y_1 \otimes Y_2$  for all  $Y_1 \in \mathcal{N}'_1$  and  $Y_2 \in \mathcal{N}'_2$ . By Theorem 3.1 again,  $\mathcal{N}'_1 \bar{\otimes} \mathcal{N}'_2$  is completely singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ .  $\square$

Combining Theorem 4.4 and Corollary 2.5, we obtain the following corollary, which generalizes Corollary 4.4 of [15].

**Corollary 4.5.** *If  $\mathcal{N}'_1$  is a singular subfactor of a type II<sub>1</sub> factor  $\mathcal{M}_1$  and  $\mathcal{N}_2$  is a completely singular von Neumann subalgebra of  $\mathcal{M}_2$ , then  $\mathcal{N}'_1 \bar{\otimes} \mathcal{N}_2$  is completely singular in  $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ .*

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